2. Έστω ότι η $z = f(x,y)$ έχει συνεχείς μερικές παραγώγους και δίνεται σε πεπλεγμένη μορφή από τη σχέση $2x^2 + 3y^2 - z^2 = 1$

Ποιος ο ρυθμός μεταβολής της z κατά την κατεύθυνση του διανύσματος $\bar{u} = 3\bar{e}_1 - 4\bar{e}_2$, εάν $f(1,1) > 0$;

Ποια η κατεύθυνση του μέγιστου ρυθμού μεταβολής της z στο $(1,1)$, εάν $f(1,1) > 0$;

Λύση: Έστω $\phi(x,y,z) = 2x^2 + 3y^2 - z^2 - 1$.

Επειδή $z = f(1,1) > 0$ έχω $\phi(x,y,z) = 0 \iff$

$z^2 = 2x^2 + 3y^2 - 1 \iff z = \pm\sqrt{2x^2+3y^2-1}$ και η αρνητική τιμή απορρίπτεται. Άρα $z = \sqrt{2x^2+3y^2-1}$ και για $(x_0, y_0) = (1,1)$ παίρνω $z_0 = \sqrt{2x_0^2 + 3y_0^2 - 1} = \sqrt{2+3-1} = 2$.

Επιπλέον,

$$\begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}} = \dfrac{2x}{z}, \text{ άρα } \dfrac{\partial z}{\partial x}(1,1) = -\dfrac{\frac{\partial \phi}{\partial x}(1,1,2)}{\frac{\partial \phi}{\partial z}(1,1,2)} = 1 \\[2ex] \dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}} = \dfrac{3y}{z}, \text{ άρα } \dfrac{\partial z}{\partial y}(1,1) = -\dfrac{\frac{\partial \phi}{\partial y}(1,1,2)}{\frac{\partial \phi}{\partial z}(1,1,2)} = \dfrac{3}{2} \end{cases}$$

Είναι, $\nabla z(1,1) = \left(\dfrac{\partial z}{\partial x}(1,1), \dfrac{\partial z}{\partial y}(1,1)\right) = \left(1, \dfrac{3}{2}\right)$

Επίσης $\bar{u}_0 = \dfrac{\bar{u}}{\|\bar{u}\|} = \dfrac{(3,-4)}{\sqrt{25}} = \left(\dfrac{3}{5}, \dfrac{-4}{5}\right)$. Εφόσον $z \in C^1$ (πολυώνυμο) στο $(1,1)$ έχω

$$\nabla_{\vec{u}_0} z(1,1) = \nabla z(1,1) \cdot \bar{u}_0 = \left(1, \frac{3}{2}\right) \cdot \left(\frac{3}{5}, \frac{-4}{5}\right) = -\frac{3}{5}.$$

Η κατεύθυνση του μέγιστου ρυθμού μεταβολής της z στο $(1,1)$ είναι η κατεύθυνση του διανύσματος κλίσης.

3. Δίνεται η συνάρτηση $z = f(x,y)$ σε πεπλεγμένη μορφή από τη σχέση $F(x,y,z(x,y)) = 0$. Αν η F έχει συνεχείς μερικές παραγώγους $2^{ης}$ τάξης, να υπολογιστούν οι ακόλουθες
$$\left\{ \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2} \right\}$$

Λύση: Κατ' αρχήν παίρνω το διαφορικό $2^{ης}$ τάξης της $F(x,y,z)$ θεωρώντας ότι οι μεταβλητές x, y είναι ανεξάρτητες, (δηλ dx, dy σταθερά), ενώ η z είναι εξαρτημένη, (δηλ dz μεταβάλλεται), και έχω:
$$d^2F = 0 \Rightarrow d(dF) = 0 \Rightarrow d\left(\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy\right.$$
$$\left. + \frac{\partial F}{\partial z} \cdot dz\right) = 0 \Rightarrow d\left(\frac{\partial F}{\partial x}\right) \cdot dx + d\left(\frac{\partial F}{\partial y}\right) \cdot dy + d\left(\frac{\partial F}{\partial z}\right) \cdot dz$$
$$+ \frac{\partial F}{\partial z} \cdot d^2 z = 0 \quad (1).$$

Υπολογίζω: $d\left(\frac{\partial F}{\partial x}\right), d\left(\frac{\partial F}{\partial y}\right), d\left(\frac{\partial F}{\partial z}\right)$

$$d\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial x^2}\cdot dx + \frac{\partial^2 F}{\partial x \partial y}\cdot dy + \frac{\partial^2 F}{\partial x \partial z}\cdot dz$$

$$= \frac{\partial^2 F}{\partial x^2}\cdot dx + \frac{\partial^2 F}{\partial x \partial y}\cdot dy + \frac{\partial^2 F}{\partial x \partial z}\cdot\left(\frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy\right)$$

$$= \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial x \partial z}\cdot\frac{\partial z}{\partial x}\right)\cdot dx + \left(\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial x \partial z}\cdot\frac{\partial z}{\partial y}\right)\cdot dy$$

$$d\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y \partial x}\cdot dx + \frac{\partial^2 F}{\partial y^2}\cdot dy + \frac{\partial^2 F}{\partial y \partial z}\cdot dz$$

$$= \frac{\partial^2 F}{\partial y \partial x}\cdot dx + \frac{\partial^2 F}{\partial y^2}\cdot dy + \frac{\partial^2 F}{\partial y \partial z}\cdot\left(\frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy\right)$$

$$= \left(\frac{\partial^2 F}{\partial y \partial x} + \frac{\partial^2 F}{\partial y \partial z}\cdot\frac{\partial z}{\partial x}\right)\cdot dx + \left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial y \partial z}\cdot\frac{\partial z}{\partial y}\right)\cdot dy$$

$$d\left(\frac{\partial F}{\partial z}\right) = \frac{\partial^2 F}{\partial z \partial x}\cdot dx + \frac{\partial^2 F}{\partial z \partial y}\cdot dy + \frac{\partial^2 F}{\partial z^2}\cdot dz$$

$$= \frac{\partial^2 F}{\partial z \partial x}\cdot dx + \frac{\partial^2 F}{\partial z \partial y}\cdot dy + \frac{\partial^2 F}{\partial z^2}\cdot\left(\frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy\right)$$

$$= \left(\frac{\partial^2 F}{\partial z \partial x} + \frac{\partial^2 F}{\partial z^2}\cdot\frac{\partial z}{\partial x}\right)\cdot dx + \left(\frac{\partial^2 F}{\partial z \partial y} + \frac{\partial^2 F}{\partial z^2}\cdot\frac{\partial z}{\partial y}\right)\cdot dy$$

Τέλος, γνωρίζω ότι:

$$
\begin{cases}
dz = \dfrac{\partial z}{\partial x} \cdot dx + \dfrac{\partial z}{\partial y} \cdot dy \\[2ex]
d^2 z = d(dz) = \dfrac{\partial^2 z}{\partial x^2} \cdot (dx)^2 + 2 \dfrac{\partial^2 z}{\partial x \partial y} \cdot dx dy + \dfrac{\partial^2 z}{\partial y^2} \cdot (dy)^2
\end{cases}
$$

Με αντικατάσταση των διαφορικών $d\left(\frac{\partial F}{\partial x}\right)$, $d\left(\frac{\partial F}{\partial y}\right)$, $d\left(\frac{\partial F}{\partial z}\right)$, dz και $d^2 z$ στην (1), έχω

$$
\begin{cases}
\dfrac{\partial^2 F}{\partial x^2} + 2 \dfrac{\partial^2 F}{\partial x \partial z} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial^2 F}{\partial z^2} \cdot \left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial^2 z}{\partial x^2} = 0 \\[2ex]
\dfrac{\partial^2 F}{\partial x \partial y} + \dfrac{\partial^2 F}{\partial x \partial z} \cdot \dfrac{\partial z}{\partial y} + \dfrac{\partial^2 F}{\partial y \partial z} \cdot \dfrac{\partial z}{\partial x} + \dfrac{\partial^2 F}{\partial z^2} \cdot \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial z}{\partial y} + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial^2 z}{\partial x \partial y} = 0 \\[2ex]
\dfrac{\partial^2 F}{\partial y^2} + 2 \dfrac{\partial^2 F}{\partial y \partial z} \cdot \dfrac{\partial z}{\partial y} + \dfrac{\partial^2 F}{\partial z^2} \cdot \left(\dfrac{\partial z}{\partial y}\right)^2 + \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial^2 z}{\partial y^2} = 0
\end{cases}
$$

Λύνω